

Note on pre-Courant algebroid structures for parabolic geometries

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Abstract

This note aims to demonstrate that every parabolic geometry has a naturally defined per-Courant algebroid structure. If the geometry is regular, this structure is a Courant algebroid if and only if the the curvature κ of the Cartan connection vanishes.

This note assumes familiarity with both parabolic geometry, and Courant algebroids. See [ČS00] and [ČG02] for a good introduction to the first case, and [KS05] and [Vai05] for the second. Some of the basic definitions will be recalled here:

Definition 0.1 (Parabolic Geometry). Let G be a semi-simple Lie group with Lie algebra \mathfrak{g} , and P a parabolic subgroup with Lie algebra \mathfrak{p} . A parabolic geometry on a manifold M is given by a principal P bundle \mathcal{P} , an inclusion $\mathcal{P} \subset \mathcal{G}$, and a principal connection $\vec{\omega}$ on \mathcal{G} . This connection is required to satisfy the condition that $\vec{\omega}|_{\mathcal{P}}$ is a linear isomorphism $T\mathcal{P} \rightarrow \mathfrak{g}$.

Let $\mathcal{A} = \mathcal{P} \times_P \mathfrak{g} = \mathcal{G} \times_G \mathfrak{g}$ and denote by $\vec{\nabla}$ the affine connection on \mathcal{A} coming from $\vec{\omega}$. By construction, \mathcal{A} also inherits an algebraic bracket $\{, \}$ and the Killing form B . It moreover has a well defined subbundle $\mathcal{A}_{(0)} = \mathcal{P} \times_P \mathfrak{p}$, and the properties of $\vec{\nabla}$ give an equivalence $\mathcal{A}/\mathcal{A}_{(0)} \cong T$, thus a projection $\pi : \mathcal{A} \rightarrow T$. The Killing form B then defines an inclusion $T^* \subset \mathcal{A}$, with $(T^*)^\perp = \mathcal{A}_{(0)}$. This implies that for v a one form, x any section of \mathcal{A} :

$$B(v, x) = v \lrcorner \pi(x).$$

Let x, y be sections of \mathcal{A} . By the above inclusion, we may see $\vec{\nabla}x$ as section of $T^* \otimes \mathcal{A} \subset \mathcal{A} \otimes \mathcal{A}$, and thus we can directly write expressions like $\vec{\nabla}_x y$ (which is equal to $\vec{\nabla}_{\pi(x)} y$ in more traditional notation). The curvature of $\vec{\nabla}$ is κ ; by inclusion, we can see this as a section of $\wedge^2 \mathcal{A} \otimes \mathcal{A}$. The parabolic structure gives a filtration on \mathcal{A} and hence a concept of minimum homogeneity for sections of any associated bundle. There is also a Lie algebra co-differential $\partial^* : \wedge^2 \mathfrak{p}^\perp \otimes \mathfrak{g} \rightarrow \mathfrak{p}^\perp \otimes \mathfrak{g}$.

Definition 0.2. A parabolic geometry is *regular* if $\text{hom}(\kappa) \geq 1$, and is normal if $\partial^* \kappa = 0$.

Drawing on the definition of [KS05]:

Definition 0.3 (Courant algebroid). A Courant algebroid is vector bundle $E \rightarrow M$, with a pseudo-Riemannian metric B on it, a projection $\pi : E \rightarrow TM$ called an anchor, and an inclusion $T^* \subset E$. It has a differential, \mathbb{R} -linear bracket $[\cdot, \cdot] : \Gamma(E) \otimes \Gamma(E) \rightarrow \Gamma(E)$. This is required to obey the following properties, for sections x, y, z of E :

1. The Jacobi identity $\mathcal{J}(x, y, z) = [x, [y, z]] - [[x, y], z] - [y, [x, z]] = 0$.
2. $\pi(x) \cdot B(y, y) = 2B(x, [y, y])$.
3. $\pi(x) \cdot B(y, z) = B([x, y], z) + B([x, z], y)$.

Note that property 2 implies that $[\cdot, \cdot]$ is not a skew bracket. A pre-Courant algebroid, as defined in [Vai05], is a structure that obeys all the previous conditions, except for property 1. Instead, it is required to simply to have a linear Jacobian $\mathcal{J} \in \Gamma(\wedge^3 \mathcal{A} \otimes \mathcal{A})$.

The point of this note is:

Theorem 0.4. *Let $(M, \mathcal{A}, \vec{\nabla})$ be a parabolic geometry. Then \mathcal{A} is a pre-Courant algebroid for a natural choice of bracket $[\cdot, \cdot]$. If $\vec{\nabla}$ is flat, then it is a Courant algebroid. If the geometry is regular, then the Jacobian of $[\cdot, \cdot]$ has the same homogeneity as the curvature – in particular, it is not a Courant algebroid whenever $\kappa \neq 0$. into a Courant algebroid.*

First, it is easy to see that \mathcal{A} has all the required algebraic properties of a Courant algebroid: a projection to T , an inclusion of T^* , and a metric coming from the Killing form B .

Paper [ČG02] defines a differential bracket $\langle \cdot, \cdot \rangle$ defined on \mathcal{A} as:

$$\langle x, y \rangle = \vec{\nabla}_x y - \vec{\nabla}_y x - \{x, y\} - \kappa(x, y).$$

Because of its original definition (defined as the bracket of right-invariant vector fields on \mathcal{P}), it must obey the Jacobi identity. This allows us to construct the (non-skew) bracket:

$$[x, y] = \langle x, y \rangle + B(y, \kappa(x, -)) - B(x, \kappa(y, -)) + B(\vec{\nabla} x, y).$$

The last term is the contraction of the second component of $\vec{\nabla} x$ with y ; it is thus always a section of T^* . Most of the remaining properties of the pre-Courant algebroid are not hard to show (paper [Vai05] demonstrates them directly). For instance:

$$[x, x] = B(\vec{\nabla} x, x) = \frac{1}{2} dB(x, x), \tag{1}$$

is immediate, (implying that $[x, y] = -[y, x] + d(B(x, y))$, while

$$\begin{aligned} B([x, y], z) + B(y, [x, z]) &= B(\vec{\nabla}_x y - \vec{\nabla}_y x - \{x, y\} - \kappa(x, y) + B(\vec{\nabla} x, y), z) \\ &\quad B(y, \vec{\nabla}_x z - \vec{\nabla}_z x - \{x, z\} - \kappa(x, z) + B(\vec{\nabla} x, z)) \\ &\quad B(B(y, \kappa(x, -), z) - B(B(x, \kappa(y, -), z) \\ &\quad + B(B(z, \kappa(x, -), y) - B(B(x, \kappa(y, -), z) \\ &= B(\vec{\nabla}_x y, z) - B(\vec{\nabla}_y x, z) + B(\vec{\nabla}_z x, y) \\ &\quad + B(\vec{\nabla}_x z, y) - B(\vec{\nabla}_z x, y) + B(\vec{\nabla}_y x, y) \\ &\quad - B(\kappa(x, y), z) - B(\kappa(x, z), y) + B(y, \kappa(x, z)) + B(z, \kappa(x, y)) \\ &\quad - B(x, \kappa(y, z)) - B(x, \kappa(z, y)) \\ &= \pi(x) \cdot B(y, z) + 0 \end{aligned}$$

gives

$$\pi(x) \cdot B(y, z) = B([x, y], z) + B(y, [x, z]). \tag{2}$$

Finally:

Proposition 0.5. *If $\mathcal{J} = [x, [y, z]] - [[x, y], z] - [y, [x, z]]$ is the Jacobian, then it is a section of $\wedge^3 \mathcal{A} \otimes \mathcal{A}$.*

Proof. First, we need to note that for any function $f \in C^\infty(M)$, $[df, y] = 0$. This follows from the fact that $[df, y]$ is clearly a one-form (the only ambiguous term is $\vec{\nabla}_y df - \{y, df\}$, which is a one-form as the

negative homogeneity components of $\vec{\nabla}_y$ and $\{y, -\}$ are the same) and from:

$$\begin{aligned}
[df, y](Z) &= (-\vec{\nabla}_y df + \{y, df\} + B(\vec{\nabla} df, y) - B(df, \kappa(y, -)))(Z) \\
&= -(\nabla df)(y, Z) - B(\{P(Y), df\}, Z) + B(\{A + v, df\}, Z) + \\
&\quad (\nabla df)(Z, y) + B(\{Z, df\}, y) + B(\{P(Z), df\}, y) - B(df, \kappa(y, Z)) \\
&= d(df)(Z, y) - df(\nabla_Z Y - \nabla_Y Z - [Z, Y]) - B(df, \kappa(y, Z)) \\
&\quad + B(df, \{P(Y), Z\} - \{P(Z), y\} - \{A + v, Z\} + \{y, Z\}) \\
&= B(df, (\{P(Y), Z\} - \{P(Z), Y\} + \{Y, Z\} + \nabla_Y Z - \nabla_Z Y - [Y, Z])_-) \\
&\quad - B(df, \kappa(y, Z)) \\
&= B(df, \kappa(Y, Z)_-) - B(df, \kappa(y, Z)) \\
&= 0,
\end{aligned}$$

where Z is any section of T , and we have used a preferred connection ∇ (see [ČG02]) to give a splitting of $\vec{\nabla}_X = X + \nabla_X + P(X)$ and $y = Y + A + v$.

This allows us to calculate:

$$\begin{aligned}
\mathcal{J}(fx, y, z) &= [fx, [y, z]] - [[fx, y], z] - [y, [fx, z]] \\
&= f[x, [y, z]] - ([y, z] \cdot f)x + B(x, [y, z])df \\
&\quad - [f[x, y], z] - [-(y \cdot f)x, z] - [B(x, y)df, z] \\
&\quad - [y, f[x, z]] + [y, (z \cdot f)x] - [y, B(z, x)df] \\
&= f\mathcal{J}(x, y, z) - ([y, z] \cdot f)x + B(x, [y, z])df \\
&\quad + (z \cdot f)[x, y] - B([x, y], z)df + (y \cdot f)[x, z] - (z \cdot y \cdot f)x + B(x, z)d(y \cdot f) \\
&\quad - B(x, y)[df, z] + (z \cdot B(x, y))df - B(df, z)dB(x, y) \\
&\quad - (y \cdot f)[x, z] + (z \cdot f)[y, x] + (y \cdot z \cdot f)x - B(z, x)[y, df] - (y \cdot B(z, x))df.
\end{aligned}$$

Re-arranging, and using the equations (1) and (2) extensively gives:

$$\begin{aligned}
&= f\mathcal{J}(x, y, z) + B(x, z)(d(y \cdot f) - [y, df]) - B(x, y)[df, z] \\
&= f\mathcal{J}(x, y, z) + B(x, z)[df, y] - B(x, y)[df, z] \\
&= f\mathcal{J}(x, y, z).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{J}(x, y, z) &= [x, [y, z]] - [[x, y], z] - [y, [x, z]] \\
&= -[x, [z, y]] + [z, [x, y]] + [[x, z], y] \\
&\quad + [x, dB(y, z)] - dB(z, [x, y]) - dB(y, [x, z]) \\
&= -\mathcal{J}(x, z, y) + [x, dB(y, z)] - d(x \cdot B(y, z)) \\
&= -\mathcal{J}(x, z, y) - [dB(y, z), x] \\
&= -\mathcal{J}(x, z, y).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{J}(x, y, z) &= [x, [y, z]] - [[x, y], z] - [y, [x, z]] \\
&= -[y, [x, z]] + ([y, x], z] + [dB(y, x), z] + [x, [y, z]] \\
&= -\mathcal{J}(y, x, z).
\end{aligned}$$

So \mathcal{J} is totally skew, and $C^\infty(M)$ -linear in the first entry, hence in every entry. \square

Given the previous result, to actually calculate \mathcal{J} at a point, it suffices to choose a particular frame at that point. Let $\{e_j\}$ be a local frame of \mathcal{A} around $p \in M$ chosen so that $(\vec{\nabla} e_j)_p = 0$. Then it is immediately evident that:

$$[e_2, e_3]_p = (\{e_2, e_3\} + \kappa(e_2, e_3) + B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3)))_p.$$

Now consider $[e_1, [e_2, e_3]]_p$. The terms in that expression will be either second derivatives of the e_j , or linear terms. This gives us, at p :

$$\begin{aligned} [e_1, [e_2, e_3]] &= \langle e_1, \langle e_2, e_3 \rangle \rangle + B(e_3, (\vec{\nabla}_{e_1} \kappa)(e_2)) - B(e_2, (\vec{\nabla}_{e_1} \kappa)(e_3)) \\ &\quad - \{e_1, B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3))\} + B(B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3)), \kappa(e_1)) \\ &\quad + B(\vec{\nabla}_{e_1}(\vec{\nabla} e_2), e_3) \end{aligned}$$

$$\begin{aligned} [e_2, [e_1, e_3]] &= \langle e_2, \langle e_1, e_3 \rangle \rangle + B(e_3, (\vec{\nabla}_{e_2} \kappa)(e_1)) - B(e_1, (\vec{\nabla}_{e_2} \kappa)(e_3)) \\ &\quad - \{e_2, B(e_3, \kappa(e_1)) - B(e_1, \kappa(e_3))\} + B(B(e_3, \kappa(e_1)) - B(e_1, \kappa(e_3)), \kappa(e_2)) \\ &\quad + B(\vec{\nabla}_{e_2}(\vec{\nabla} e_1), e_3) \end{aligned}$$

$$\begin{aligned} [[e_1, e_2], e_3] &= -[e_3, [e_1, e_2]] + dB(e_3, [e_1, e_2]) \\ &= -\langle e_3, \langle e_1, e_2 \rangle \rangle - B(e_2, (\vec{\nabla}_{e_3} \kappa)(e_1)) + B(e_1, (\vec{\nabla}_{e_3} \kappa)(e_2)) \\ &\quad + \{e_3, B(e_2, \kappa(e_1)) - B(e_1, \kappa(e_2))\} - B(B(e_2, \kappa(e_1)) - B(e_1, \kappa(e_2)), \kappa(e_3)) \\ &\quad - B(\vec{\nabla}_{e_3}(\vec{\nabla} e_1), e_2) \\ &\quad + B(e_3, \vec{\nabla}(\vec{\nabla}_{e_1} e_2 - \vec{\nabla}_{e_2} e_1) - (\vec{\nabla} \kappa)(e_1, e_2) + B(e_2, (\vec{\nabla} \kappa)(e_1)) - B(e_1, (\vec{\nabla} \kappa)(e_2))) \\ &\quad + B(e_3, B(\vec{\nabla}(\vec{\nabla} e_1), e_2)) \end{aligned}$$

Now \langle, \rangle obeys the Jacobi identity. We get further simplifications of the type $(\vec{\nabla}_{e_1} \kappa)(e_2) = (\vec{\nabla}_{e_2} \kappa)(e_1) - (\vec{\nabla} \kappa)(e_1, e_2)$ by the Bianci identity on $\vec{\nabla}$. Remembering the identity $B(e_3, v) = v(e_3)$ for any one-form v gives simplifications in the $[[e_1, e_2], e_3]$ term. Together, these give the relation:

$$\begin{aligned} \mathcal{J}(e_1, e_2, e_3) &= -\{e_1, B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3))\} + B(e_3, \kappa(e_2, \kappa(e_1))) - B(e_2, \kappa(e_3, \kappa(e_1))) \\ &\quad + \text{cyclic terms.} \end{aligned}$$

This demonstrates that if $\kappa = 0$, then we are in the presence of a Courant algebroid. However, if $\kappa \neq 0$ and $\vec{\nabla}$ is regular, this construction will always fail to be a Courant algebroid:

Proposition 0.6. *If the Cartan connection is regular and $\kappa \neq 0$ at a point then $\mathcal{J} \neq 0$ at that point, and $\text{hom}(\mathcal{J}) = \text{hom}(\kappa)$.*

Proof. Assume $\kappa \neq 0$ at p . Let $h = \text{hom}(\kappa)$. Since κ is regular, $h > 0$. Since B and $\{, \}$ preserve homogeneity,

$$\text{hom}(\mathcal{J}) \geq h.$$

At p , let us project κ onto its lowest homogeneity component κ_H (if κ is normal, this is just the lowest homogeneity harmonic curvature [Čap06]). This κ_H can further be decomposed into sections $\kappa_{a,b,c}$ of $T_a^* \wedge T_b^* \otimes \mathcal{A}_c$ for integers a, b and c , with $a + b + c = h$. Pick a, b and c such that $\kappa_{a,b,c} \neq 0$ at p .

The terms in \mathcal{J} with two appearances of κ are of homogeneity $\geq 2h > h$, so we will ignore them. Chose a Weyl structure ∇ that preserve a volume form to give a splitting $\mathcal{A} = \sum_{i=-k}^k \mathcal{A}_i$. The Killing

form B ensures that $\mathcal{A}_i \perp \mathcal{A}_j$ whenever $j \neq -i$. Define $\mathcal{A}_{(j)} = \sum_{i=j}^k \mathcal{A}_i$ (this does not depend on the choice of ∇).

Let E be the grading section in \mathcal{A}_0 , X_{-a} a section of $\mathcal{A}_{-a} = T_{-a}$ and Y_{-c} a section of \mathcal{A}_{-c} . Call \mathcal{J}_h the homogeneity h component of \mathcal{J} . Now $\kappa_H(E) = 0$, and assume for the moment that $c \neq 0$, implying that $B(E, \kappa_H(X_{-a}))$ has trivial projection onto the T_b^* factor. We will now look, in homogeneity h , at the T_b^* factor of $\mathcal{J}(E, X_{-a}, Y_{-c})$:

$$\begin{aligned} & \left(\{E, B(Y_{-c}, \kappa_H(X_{-a}))\} - \{E, B(X_{-a}, \kappa_H(Y_{-c}))\} \right) \\ &= \\ & \{E, (Id - m)(\kappa_H)(X_{-a}, -, Y_{-c})\}, \end{aligned}$$

where m is the operator interchanging the first and last component of $\otimes^3 \mathcal{A}$. Basic representation theory implies that $Id - m$ is injective on $\wedge^2 \mathcal{A} \otimes \mathcal{A}$, hence on κ . Moreover $Id - m$ preserves homogeneity, so there must exist X_{-a}, Y_{-c} such that $(Id - m)(\kappa_H)(X_{-a}, -, Y_{-c})/(\mathcal{A}_{(b+1)})$ is a non-zero section of \mathcal{A}_b around p . The bracket with E does not change this, as E acts by multiplication by b on these sections, and $b > 0$ since κ is a curvature.

Now if $c = 0$, then let Z_0 be a section of \mathcal{A}_0 , and then

$$(\mathcal{J})_b(X_{-a}, Y_0, Z_0) = (Id - m)(\phi \circ \kappa_H(X_{-a}))(Y_0, Z_0).$$

Here ϕ is the map $\mathcal{A} \rightarrow \wedge^2 \mathcal{A}$ given by the Lie bracket. Since \mathcal{A} is semi-simple, ϕ has no kernel, making $\phi \circ \kappa_H(X_{-a})$ into a non-degenerate section of $\wedge^2 \mathcal{A} \otimes \mathcal{A}$. But $Id - m$ is injective on this bundle, ensuring that there must exist Y_0 and Z_0 making the above expression non-zero.

This demonstrates that $\mathcal{J} \neq 0$ whenever $\kappa \neq 0$ and further, that $\text{hom}(\mathcal{J}) = h$.

□

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